To bring these magical squares to the surface the squares of each set of parallel squares may be permuted as follows:

Original order 1, 2, 3, 4, 5, 6,
Permuted order ....3, 2, 1, 6, 5, 4.

The result is the final cube shown in the beginning of this article.

The above permutation is subject to two conditions. The several sets of parallel squares must all be permuted in the same manner. Any two parallel squares which in the original cube are located on opposite sides of the middle plane of the cube and at an equal distance from it, in the permuted cube must be located on opposite sides of the middle plane of the cube and at an equal distance from it. These conditions are for the protection of the diagonals.

JOHN WORTHINGTON.

MAGIC IN THE FOURTH DIMENSION.

Definition of terms: *Row* is a general term; *rank* denotes a horizontal right-to-left row; *file* a row from front to back; and *column* a vertical row in a cube—not used of any horizontal dimension.

If n^2 numbers of a given series can be grouped so as to form a magic square and n such squares be so placed as to constitute a magic cube, why may we not go a step further and group n cubes in relations of the fourth dimension? In a magic square containing the natural series $1 \ldots n^2$ the summation is $\frac{n(n^2+1)}{2}$; in a magic cube with the series $1 \ldots n^3$ it is $\frac{n(n^3+1)}{2}$; and in an analogous fourth-dimension construction it naturally will be $\frac{n(n^4+1)}{2}$.

With this idea in mind I have made some experiments, and the results are interesting. The analogy with squares and cubes is not perfect, for rows of numbers can be arranged side by side to represent a visible square, squares can be piled one upon another to make a visible cube, but cubes cannot be so combined in drawing as to picture to the eye their higher relations. My expectation *a priori* was that some connection or relation, probably through some form of diagonal-of-diagonal, would be found to exist between the cubes containing the n^4 terms of a series. This particular feature did appear in the cases where n was odd. Here is how it worked out:

I. When n is odd.

1. Let $n=3$, then S=123.—The natural series 1...81 was divided into three sub-series such that the sum of each would be

one-third the sum of the whole. In dealing with any such series when n is odd there will be n sub-series, each starting with one of the first n numbers, and the difference between successive terms will be $n+1$, except after a multiple of n, when the difference is 1. In the present case the three sub-series begin respectively with 1, 2, 3, and the first is 1 5 9 10 14 18 19 23 27 28 32 36 37 41 45 46 50 54 55 59 63 64 68 72 73 77 81. These numbers were arranged in three squares constituting a magic cube, and the row of squares so formed was flanked on right and left by similar rows formed from the other two sub-series (see Fig. 1).

It is not easy—perhaps it is not possible—to make an absolutely perfect cube of 3. These are not perfect, yet they have many

I			II			III		
25	38	60	28	77	18	67	8	48
33	79	11	72	1	50	21	40	62
65	6	52	23	45	55	35	75	13
29	78	16	68	9	46	26	39	58
70	2	51	19	41	63	31	80	12
24	43	56	36	73	14	66	4	53
69	7	47	27	37	59	30	76	17
20	42	61	32	81	10	71	3	49
34	74	15	64	5	54	22	44	57

Fig. 1. (3^4)

striking features. Taking the three cubes separately we find that in each all the "straight" dimensions—rank, file and column—have the proper footing, 123. In the middle cube there are two plane diagonals having the same summation, and in cubes I and III one each. In cube II four cubic diagonals and four diagonals of vertical squares are correct; I and III each have one cubic diagonal and one vertical-square diagonal.

So much for the original cubes; now for some combinations. The three squares on the diagonal running down from left to right will make a magic cube with rank, file, column, cubic diagonals, two plane diagonals and four vertical-square diagonals (37 in all) correct. Two other cubes can be formed by starting with the top squares of II and III respectively and following the "broken diag-

onals" running downward to the right. In each of these S occurs at least 28 times (in 9 ranks, 9 files, 9 columns and one cubic diagonal). Various other combinations may be found by taking the squares together in horizontal rows and noting how some columns and assorted diagonals have the proper summation, but the most important and significant are those already pointed out. In all the sum 123 occurs over 200 times in this small figure.

		I					II					III					IV					V		
317	473	604	10	161	192	348	479	510	36	67	223	354	385	536	567	98	229	260	411	442	598	104	135	286
110	136	292	448	579	610	11	167	323	454	485	511	42	198	329	360	386	542	73	204	235	261	417	573	79
423	554	85	236	267	298	429	585	111	142	173	304	460	611	17	48	179	335	486	517	548	54	210	361	392
211	367	398	529	60	86	242	273	404	560	586	117	148	279	435	461	617	23	154	310	336	492	523	29	185
504	35	186	342	498	379	535	61	217	373	254	410	561	92	248	129	285	436	592	123	4	160	311	467	623
606	12	168	324	455	481	512	43	199	330	356	387	543	74	205	231	262	418	574	80	106	137	293	449	580
299	430	581	112	143	174	305	456	612	18	49	180	331	487	518	549	55	206	362	393	424	555	81	237	268
87	243	274	405	556	587	118	149	280	431	462	618	24	155	306	337	493	524	30	181	212	368	399	530	56
380	531	62	218	374	255	406	562	93	249	130	281	437	593	124	5	156	312	468	624	505	31	187	343	499
193	349	480	506	37	68	224	355	381	537	568	99	230	256	412	443	599	105	131	287	318	474	605	6	162
175	301	457	613	19	50	176	332	488	519	550	51	207	363	394	425	551	82	238	269	300	426	582	113	144
588	119	150	276	432	463	619	25	151	397	338	494	525	26	182	213	369	400	526	57	88	244	275	401	557
251	407	563	94	250	126	282	438	594	125	1	157	313	469	625	501	32	188	344	500	376	532	63	219	375
69	225	351	382	538	569	100	226	257	413	444	600	101	132	288	319	475	601	7	163	194	350	476	507	38
482	513	44	200	326	357	388	544	75	201	232	263	419	575	76	107	138	294	450	576	607	13	169	325	451
464	620	21	152	308	339	495	521	27	183	214	370	396	527	58	89	245	271	402	558	589	120	146	277	433
127	283	439	595	121	2	158	314	470	621	502	33	189	345	496	377	533	64	220	371	252	408	564	95	246
570	96	227	258	414	445	596	102	133	289	320	471	602	8	164	195	346	477	508	39	70	221	352	383	539
358	389	545	71	202	233	264	420	571	77	108	139	295	446	577	608	14	170	321	452	483	514	45	196	327
46	177	333	489	520	546	52	208	364	395	421	552	83	239	270	296	427	583	114	145	171	302	458	614	20
3	159	315	466	622	503	34	190	341	497	378	534	65	216	372	253	409	565	91	247	128	284	440	591	122
441	597	103	134	290	316	472	603	9	165	191	347	478	509	40	66	222	353	384	540	566	97	288	259	415
234	265	416	572	78	109	140	291	447	578	609	15	166	322	453	484	515	41	197	328	359	390	541	72	203
547	53	209	365	391	422	553	84	240	266	297	428	584	115	141	172	303	459	615	16	47	178	334	490	516
340	491	522	28	184	215	366	397	528	59	90	241	272	403	559	590	116	147	278	434	465	616	22	153	309

Fig. 2. (5^4)

One most interesting fact remains to be noticed. While the three cubes were constructed separately and independently the figure formed by combining them is an absolutely perfect square of 9, with a summation of 369 in rank, file and corner diagonal (besides all "broken" diagonals running downward to the right), and a perfect

balancing of complementary numbers about the center. Any such pair, taken with the central number 41, gives us the familiar sum 123, and this serves to bind the whole together in a remarkable manner.

2. Let $n=5$, then S=1565.—In Fig. 2 is represented a group of 5-cubes each made up of the numbers in a sub-series of the natural series 1...625. In accordance with the principle stated in a previous paragraph the central sub-series is 1 7 13 19 25 26 32 ... 625, and the other four can easily be discovered by inspection. Each of the twenty-five small squares has the summation 1565 in rank, file, corner diagonal and broken diagonals, twenty times altogether in each square, or 500 times for all.

Combining the five squares in col. I we have a cube in which all the 75 "straight" rows (rank, file and vertical column), all the horizontal diagonals and three of the four cubic diagonals foot up 1565. In cube III all the cubic diagonals are correct. Each cube also has seven vertical-square diagonals with the same summation. Taking together the squares in horizontal rows we find certain diagonals having the same sum, but the columns do not. The five squares in either diagonal of the large square, however, combine to produce almost perfect cubes, with rank, file, column and cubic diagonals all correct, and many diagonals of vertical squares.

A still more remarkable fact is that the squares in the broken diagonals running in either direction also combine to produce cubes as nearly perfect as those first considered. Indeed, the great square seems to be an enlarged copy of the small squares, and where the cells in the small ones unite to produce S the corresponding squares in the large figure unite to produce cubes more or less perfect. Many other combinations are discoverable, but these are sufficient to illustrate the principle, and show the interrelations of the cubes and their constituent squares. The summation 1565 occurs in this figure not less than 1400 times.

The plane figure containing the five cubes (or twenty-five squares) is itself a perfect square with a summation of 7825 for every rank, file, corner or broken diagonal. Furthermore all complementary pairs are balanced about the center, as in Fig. 1. Any square group of four, nine or sixteen of the small squares is magic, and if the group of nine is taken at the center it is "perfect." It is worthy of notice that all the powers of n above the first lie in the middle rank of squares, and that all other multiples of n are grouped in regular relations in the other ranks and have the same

grouping in all the squares of any given rank. The same is true of the figure illustrating 7^4, which is to be considered next.

3. Let $n=7$, then $S=8407$.—This is so similar in all its properties to the 5-construction just discussed that it hardly needs separate description. It is more nearly perfect in all its parts than the 5^4, having a larger proportion of its vertical-square diagonals correct. Any square group of four, nine, sixteen, twenty-five or thirty-six small squares is magic, and if the group of nine or twenty-five

I				II				III				IV			
1	255	254	4	248	10	11	245	240	18	19	237	25	231	230	28
252	6	7	249	13	243	242	16	21	235	234	24	228	30	31	225
8	250	251	5	241	15	14	244	233	23	22	236	32	226	227	29
253	3	2	256	12	246	247	9	20	238	239	17	229	27	26	232
224	34	35	221	41	215	214	44	49	207	206	52	200	58	59	197
37	219	218	40	212	46	47	209	204	54	55	201	61	195	194	64
217	39	38	220	48	210	211	45	56	202	203	53	193	63	62	196
36	222	223	33	213	43	42	216	205	51	50	208	60	198	199	57
192	66	67	189	73	183	182	76	81	175	174	84	168	90	91	165
69	187	186	72	180	78	79	177	172	86	87	169	93	163	162	96
185	71	70	188	80	178	179	77	88	170	171	85	161	95	94	164
68	190	191	65	181	75	74	184	173	83	82	176	92	166	167	89
97	159	158	100	152	106	107	149	144	114	115	141	121	135	134	124
156	102	103	153	109	147	146	112	117	139	138	120	132	126	127	129
104	154	155	101	145	111	110	148	137	119	118	140	128	130	131	125
157	99	98	160	108	150	151	105	116	142	143	113	133	123	122	136

Fig. 3. (4^4)

be taken at the center of the figure it is "perfect." The grouping of multiples and powers of n is very similar to that already described for 5^4.

II. When n is even.

I. Let $n=4$, then $S=514$.—The numbers may be arranged in either of two ways. If we take the diagram for the 4-cube as

I						II						III					
1	1295	1294	3	1292	6	1278	20	21	1276	23	1273	37	1259	1258	39	1256	42
1290	8	1288	1287	11	7	25	1271	27	28	1268	1272	1254	44	1252	1251	47	43
1284	1283	15	16	14	1279	31	32	1264	1263	1265	36	1248	1247	51	52	50	1243
13	17	1281	1282	1280	18	1266	1262	34	33	35	1261	49	53	1245	1246	1244	54
12	1286	9	10	1289	1285	1267	29	1270	1269	26	30	48	1250	45	46	1253	1249
1291	2	4	1293	5	1296	24	1277	1275	22	1274	19	1255	38	40	1257	41	1260
1188	110	111	1186	113	1183	127	1169	1168	129	1166	132	1152	146	147	1150	149	1147
115	1181	117	118	1178	1182	1164	134	1162	1161	137	133	151	1145	153	154	1142	1146
121	122	1174	1173	1175	126	1158	1157	141	142	140	1153	157	158	1138	1137	1139	162
1176	1172	124	123	125	1171	139	143	1155	1156	1154	144	1140	1136	160	159	161	1135
1177	119	1180	1179	116	120	138	1160	135	136	1163	1159	1141	155	1144	1143	152	156
114	1187	1185	112	1184	109	1165	128	130	1167	131	1170	150	1151	1149	148	1148	145
217	1079	1078	219	1076	222	1062	236	237	1060	239	1057	253	1043	1042	255	1040	258
1074	224	1072	1071	227	223	241	1055	243	244	1052	1056	1038	260	1036	1035	263	259
1068	1067	231	232	230	1063	247	248	1048	1047	1049	252	1032	1031	267	268	266	1027
229	233	1065	1066	1064	234	1050	1046	250	249	251	1045	265	269	1029	1030	1028	270
228	1070	225	226	1073	1069	1051	245	1054	1053	242	246	264	1034	261	262	1037	1033
1075	218	220	1077	221	1080	240	1061	1059	238	1058	235	1039	254	256	1041	257	1044
865	431	430	867	428	870	414	884	885	412	887	409	901	395	394	903	392	906
426	872	424	423	875	871	889	407	891	892	404	408	390	908	388	387	911	907
420	419	879	880	878	415	895	896	400	399	401	900	384	383	915	916	914	379
877	881	417	418	416	882	402	398	898	897	899	397	913	917	381	382	380	918
876	422	873	874	425	421	403	893	406	405	890	894	912	386	909	910	389	385
427	866	868	429	869	432	888	413	411	886	410	883	391	902	904	393	905	396
864	434	435	862	437	859	451	845	844	453	842	456	828	470	471	826	473	823
439	857	441	442	854	858	840	458	838	837	461	457	475	821	477	478	818	822
445	446	850	849	851	450	834	833	465	466	464	829	481	482	814	813	815	486
852	848	448	447	449	847	463	467	831	832	830	468	816	812	484	483	485	811
853	443	856	855	440	444	462	836	459	460	839	835	817	479	820	819	476	480
438	863	861	436	860	433	841	452	454	843	455	846	474	827	825	472	824	469
756	542	543	754	545	751	559	737	736	561	734	564	720	578	579	718	581	715
547	749	549	550	746	750	732	566	730	729	569	565	583	713	585	586	710	714
553	554	742	741	743	558	726	725	573	574	572	721	589	590	706	705	707	594
744	740	556	555	557	739	571	575	723	724	722	576	708	704	592	591	593	703
745	551	748	747	548	552	570	728	567	568	731	727	709	587	712	711	584	588
546	755	753	544	752	541	733	560	562	735	563	738	582	719	717	580	716	577

Fig. 4, First Part. (6^4:S=3891)

IV						V						VI					
1225	71	70	1227	68	1230	1224	74	75	1222	77	1219	1206	92	93	1204	95	1201
66	1232	64	63	1235	1231	79	1217	81	82	1214	1218	97	1199	99	100	1196	1200
60	59	1239	1240	1238	55	85	86	1210	1209	1211	90	103	104	1192	1191	1193	108
1237	1241	57	58	56	1242	1212	1208	88	87	89	1207	1194	1190	106	105	107	1189
1236	62	1233	1234	65	61	1213	83	1216	1215	80	84	1195	101	1198	1197	98	102
67	1226	1228	69	1229	72	78	1223	1221	76	1220	73	96	1205	1203	94	1202	91
180	1118	1119	178	1121	175	181	1115	1114	183	1112	186	199	1097	1096	201	1094	204
1123	173	1125	1126	170	174	1110	188	1108	1107	191	187	1092	206	1090	1089	209	205
1129	1130	166	165	167	1134	1104	1103	195	196	194	1099	1086	1085	213	214	212	1081
168	164	1132	1131	1133	163	193	197	1101	1102	1100	198	211	215	1083	1084	1082	216
169	1127	172	171	1124	1128	192	1106	189	190	1109	1105	210	1088	207	208	1091	1087
1122	179	177	1120	176	1117	1111	182	184	1113	185	1116	1093	200	202	1095	203	1098
1009	287	286	1011	284	1014	1008	290	291	1006	293	1003	990	308	309	988	311	985
282	1016	280	279	1019	1015	295	1001	297	298	998	1002	313	983	315	316	980	984
276	275	1023	1024	1022	271	301	302	994	993	995	306	319	320	976	975	977	324
1021	1025	273	274	272	1026	996	992	304	303	305	991	978	974	322	321	323	973
1020	278	1017	1018	281	277	997	299	1000	999	296	300	979	317	982	981	314	318
283	1010	1012	285	1013	288	294	1007	1005	292	1004	289	312	989	987	310	986	307
361	935	934	363	932	366	360	938	939	358	941	355	342	956	957	340	959	337
930	368	928	927	371	367	943	353	945	946	350	354	961	335	963	964	332	336
924	923	375	376	374	919	949	950	346	345	347	954	967	968	328	327	329	972
373	377	921	922	920	378	348	344	952	951	953	343	330	326	970	969	971	325
372	926	369	370	929	925	349	947	352	351	944	948	331	965	334	333	962	966
931	362	364	933	365	936	942	359	357	940	356	937	960	341	339	958	338	955
504	794	795	502	797	499	505	791	790	507	788	510	523	773	772	525	770	528
799	497	801	802	494	498	786	512	784	783	515	511	768	530	766	765	533	529
805	806	490	489	491	810	780	779	519	520	518	775	762	761	537	538	536	757
492	488	808	807	809	487	517	521	777	778	776	522	535	539	759	760	758	540
493	803	496	495	800	804	516	782	513	514	785	781	534	764	531	532	767	763
798	503	501	796	500	793	787	506	508	789	509	792	769	524	526	771	527	774
612	686	687	610	689	607	613	683	682	615	680	618	631	665	664	633	662	636
691	605	693	694	602	606	678	620	676	675	623	619	660	638	658	657	641	637
697	698	598	597	599	702	672	671	627	628	626	667	654	653	645	646	644	649
600	596	700	699	701	595	625	629	669	670	668	630	643	647	651	652	650	648
601	695	604	603	692	696	624	674	621	622	677	673	642	656	639	640	659	655
690	611	609	688	608	685	679	614	616	681	617	684	661	632	634	663	635	666

Fig. 4, Second Part. (6^4:S=3891)

given in *Magic Squares and Cubes* and simply extend it to cover the larger numbers involved we shall have a group of four cubes in which all the "straight" dimensions have S=514, but no diagonals except the four cubic diagonals. Each horizontal row of squares will produce a cube having exactly the same properties as those in the four vertical rows. If the four squares in either diag-

I III

1	4095	4094	4	5	4091	4090	8	4032	66	67	4029	4028	70	71	4025
4088	10	11	4085	4084	14	15	4081	73	4023	4022	76	77	4019	4018	80
4080	18	19	4077	4076	22	23	4073	81	4015	4014	84	85	4011	4010	88
25	4071	4070	28	29	4067	4066	32	4008	90	91	4005	4004	94	95	4001
4065	31	30	4068	4069	27	26	4072	96	4002	4003	93	92	4006	4007	89
24	4074	4075	21	20	4078	4079	17	4009	87	86	4012	4013	83	82	4016
16	4082	4083	13	12	4086	4087	9	4017	79	78	4020	4021	75	74	4024
4089	7	6	4092	4093	3	2	4096	72	4026	4027	69	68	4030	4031	65
4064	34	35	4061	4060	38	39	4057	97	3999	3998	100	101	3995	3994	104
41	4055	4054	44	45	4051	4050	48	3992	106	107	3989	3988	110	111	3985
49	4047	4046	52	53	4043	4042	56	3984	114	115	3981	3980	118	119	3977
4040	58	59	4037	4036	62	63	4033	121	3975	3974	124	125	3971	3970	128
64	4034	4035	61	60	4038	4039	57	3969	127	126	3972	3973	123	122	3976
4041	55	54	4044	4045	51	50	4048	120	3978	3979	117	116	3982	3983	113
4049	47	46	4052	4053	43	42	4056	112	3986	3987	109	108	3990	3991	105
40	4058	4059	37	36	4062	4063	33	3993	103	102	3996	3997	99	98	4000

II IV

Fig. 5, 8^4, First Part (One cube written).

onal of the figure be piled together neither vertical columns nor cubic diagonals will have the correct summation, but all the diagonals of vertical squares in either direction will. Regarding the whole group of sixteen squares as a plane square we find it magic, having the summation 2056 in every rank, file and corner diagonal, 1028

in each half-rank or half-file, and 514 in each quarter-rank or quarter-file. Furthermore all complementary pairs are balanced about the center.

The alternative arrangement shown in Fig. 3 makes each of the small squares perfect in itself, with every rank, file and corner diagonal footing up 514 and complementary pairs balanced about the

V VII

3968	130	131	3965	3964	134	135	3961	193	3903	3902	196	197	3899	3898	200
137	3959	3958	140	141	3955	3954	144	3896	202	203	3893	3892	206	207	3889
145	3951	3950	148	149	3947	3946	152	3888	210	211	3885	3884	214	215	3881
3944	154	155	3941	3940	158	159	3937	217	3879	3878	220	221	3875	3874	224
160	3938	3939	157	156	3942	3943	153	3873	223	222	3876	3877	219	218	3880
3945	151	150	3948	3949	147	146	3952	216	3882	3883	213	212	3886	3887	209
3953	143	142	3956	3957	139	138	3960	208	3890	3891	205	204	3894	3895	201
136	3962	3963	133	132	3966	3967	129	3897	199	198	3900	3901	195	194	3904
161	3935	3934	164	165	3931	3930	168	3872	226	227	3869	3868	230	231	3865
3928	170	171	3925	3924	174	175	3921	233	3863	3862	236	237	3859	3858	240
3920	178	179	3917	3916	182	183	3913	241	3855	3854	244	245	3851	3850	248
185	3911	3910	188	189	3907	3906	192	3848	250	251	3845	3844	254	255	3841
3905	191	190	3908	3909	187	186	3912	256	3842	3843	253	252	3846	3847	249
184	3914	3915	181	180	3918	3919	177	3849	247	246	3852	3853	243	242	3856
176	3922	3923	173	172	3926	3927	169	3857	239	238	3860	3861	235	234	3864
3929	167	166	3932	3933	163	162	3936	232	3866	3867	229	228	3870	3871	225

VI VIII

Fig. 5, 8^4, Second Part (One cube written).

center. As in the other arrangement the squares in each vertical or horizontal row combine to make cubes whose "straight" dimensions all have the right summation. In addition the new form has the two plane diagonals of each original square (eight for each cube), but sacrifices the four cubic diagonals in each cube. In lieu

of these we find a complete set of "bent diagonals" ("Franklin") like those described for the magic cube of six in *The Monist* for July, 1909.

If the four squares in either diagonal of the large figure be piled up it will be found that neither cubic diagonal nor vertical column is correct, but that all diagonals of vertical squares facing toward front or back are. Taken as a plane figure the whole group makes up a magic square of 16 with the summation 2056 in every rank, file or corner diagonal, half that summation in half of each of those dimensions, and one-fourth of it in each quarter dimension.

2. Let $n=6$, then $S=3891$.—With the natural series 1...1296 squares were constructed which combined to produce the six magic cubes of six indicated by the Roman numerals in Fig. 4. These have all the characteristics of the 6-cube described in *The Monist* of July last—108 "straight" rows, 12 plane diagonals and 24 "bent" diagonals in each cube, with the addition of 32 vertical-square diagonals if the squares are piled in a certain order. A seventh cube with the same features is made by combining the squares in the lowest horizontal row—i. e., the bottom squares of the numbered cubes. The feature of the cubic bent diagonals is found on combining any three of the small squares, no matter in what order they are taken. In view of the recent discussion of this cube it seems unnecessary to give any further account of it now.

The whole figure, made up as it is of thirty-six magic squares, is itself a magic square of 36 with the proper summation (23346) for every rank, file and corner diagonal, and the corresponding fractional part of that for each half, third or sixth of those dimensions. Any square group of four, nine, sixteen or twenty-five of the small squares will be magic in all its dimensions.

3. Let $n=8$, then $S=16388$.—The numbers 1...4096 may be arranged in several different ways. If the diagrams in Mr. Andrews's book be adopted we have a group of eight cubes in which rank, file, column and cubic diagonal are correct (and in which the halves of these dimensions have the half summation), but all plane diagonals are irregular. If the plan be adopted of constructing the small squares of complementary couplets, as in the 6-cube, the plane diagonals are equalized at the cost of certain other features. I have used therefore a plan which combines to some extent the advantages of both the others.

It will be noticed that each of the small squares in Fig. 5 is

perfect in that it has the summation 16388 for rank, file and corner diagonal (also for broken diagonals if each of the separated parts contain two, four or six—not an odd number of cells), and in balancing complementary couplets. When the eight squares are piled one upon the other a cube results in which rank, file, column, the plane diagonals of each horizontal square, the four ordinary cubic diagonals and 32 cubic bent diagonals all have S=16388. What is still more remarkable, the *half* of each of the "straight" dimensions and of each cubic diagonal has half that sum. Indeed this cube of eight can be sliced into eight cubes of 4 in each of which every rank, file, column and cubic diagonal has the footing 8194; and each of these 4-cubes can be subdivided into eight tiny 2-cubes in each of which the eight numbers foot up 16388.

So much for the features of the single cube here presented. As a matter of fact only the one cube has actually been written out. The plan of its construction, however, is so simple and the relations of numbers so uniform in the powers of 8 that it was easy to investigate the properties of the whole 8^4 scheme without having the squares actually before me. I give here the initial number of each of the eight squares in each of the eight cubes, leaving it for some one possessed of more leisure to write them all out and verify my statements as to the intercubical features. It should be remembered that in each square the number diagonally opposite the one here given is its complement, i. e., the number which added to it will give the sum 4097.

I	II	III	IV	V	VI	VII	VIII
1	3840	3584	769	3072	1281	1537	2304
4064	289	545	3296	1057	2784	2528	1825
4032	321	577	3264	1089	2752	2496	1857
97	3744	3488	865	2976	1377	1633	2208
3968	385	641	3200	1153	2688	2432	1921
161	3680	3424	929	2912	1441	1697	2144
193	3648	3392	961	2880	1473	1729	2112
3872	481	737	3104	1249	2592	2336	2017
16388	16388	16388	16388	16388	16388	16388	16388

Each of the sixty-four numbers given above will be at the upper left-hand corner of a square and its complement at the lower right-hand corner. The footings given are for these initial numbers,

but the arrangement of numbers in the squares is such that the footing will be the same for every one of the sixty-four columns in each cube. If the numbers in each horizontal line of the table above be added they will be found to have the same sum: consequently the squares headed by them must make a cube as nearly perfect as the example given in Fig. 5, which is cube I of the table above. But the sum of half the numbers in each line is half of 16388, and hence each of the eight cubes formed by taking the squares in the horizontal rows is capable of subdivision into 4-cubes and 2-cubes, like our original cube. We thus have sixteen cubes, each with the characteristics described for the one presented in Fig. 5.

If we pile the squares lying in the diagonal of our great square (starting with 1, 289, etc., or 2304, 2528, etc.) we find that its columns and cubic diagonals are not correct; but all the diagonals of its vertical squares are so, and even here the remarkable feature of the half-dimension persists.

Of course there is nothing to prevent one's going still further and examining constructions involving the fifth or even higher powers, but the utility of such research may well be doubted. The purpose of this article is to suggest in sketch rather than to discuss exhaustively an interesting field of study for some one who may have time to develop it.

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